Calculus II: For Physical Sciences and Engineering (AS.110.109)

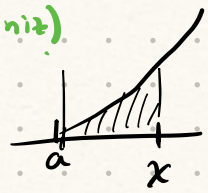
Lecture: 2

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Textbook Sections: 5.4, 5.5

At the end of the last lecture, we recalled the following:

FTC I: (Barrow, Newton, Leibniz) Let $f(x)$ be a continuous function on $[a,b]$, and for $x \in [a,b]$, define the function



$$F(x) = \int_a^x f(t)\,dt = \text{definite integral of } f \text{ from } a \text{ to the variable } x$$

Then $F(x)$ is a continuous function of x on $[a,b]$, and is a differentiable function of x on (a,b) with $F'(x) = f(x)$.

FTC II: (Barrow, Newton, Leibniz) Let $f(x)$ be a function on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is **any** anti-derivative of $f(x)$ on $[a,b]$ — i.e. any cts. function $F(x)$ on $[a,b]$ such that $F'(x) = f(x)$ on (a,b)

Today, our goal is to get a lot of practice computing integrals with these theorems, and learn some useful techniques for simplifying complicated looking problems.

Definite vs Indefinite Integrals

The FTC gives a link between the definite integral of a function and its anti-derivatives.

In particular, if $f(x)$ is a continuous function, then FTC I tells us that

$$F(x) = \int_a^x f(t)\,dt \qquad x \in [a,b]$$

is a possible anti-derivative of $f(x)$ (so is $F(x)+C$ for any constant C, since $(F(x)+C)' = F'(x) = f$).

Because of this source of anti-derivatives, we define a convenient notation for anti-derivatives in general by writing

$\int f(x)\,dx$ for any anti-derivative $F(x)$ of $f(x)$

We call $\int f(x)\,dx$ an indefinite integral of $f(x)$.

⚠ An indefinite integral represents the entire family of anti-derivatives for a function. A definite integral is always a number.

For example:

1) $f(x) = x \longrightarrow \int x\,dx = \frac{1}{2}x^2 + C \longrightarrow \frac{1}{2}x^2 + 1$

2) $f(x) = e^x \longrightarrow \int e^x\,dx = e^x + C \longrightarrow e^x$

3) $f(x) = \cos x \longrightarrow \int \cos x\,dx = \sin x + C \longrightarrow \sin x - \pi$

indefinite integrals — particular anti-derivatives

In recitation and on homework 1, we'll review some common indefinite integrals, and rules for dealing with them.

Warning: It's important to remember that any given antiderivative must be defined on a specific interval.

For example, consider the function $f(x) = \frac{1}{x^2}$, which is defined on $(-\infty, 0) \cup (0, \infty)$, but NOT at $x = 0$.

- $f(x)$ has many possible antiderivatives, such as

$$F(x) = -\frac{1}{x}, \quad F(x) = -\frac{1}{x} + 5, \quad F(x) = -\frac{1}{x} - \frac{17}{32}, \text{ etc...}$$

and each of these is also defined everywhere except at $x = 0$.

- In the most general possible case, we can write

$$F(x) = \begin{cases} -\frac{1}{x} + C_1 & \text{for all } x < 0 \\ -\frac{1}{x} + C_2 & \text{for all } x > 0 \end{cases}$$

with C_i two arbitrarily chosen constants. This is the most complete/accurate/general way to write an anti-derivative out!

⚠ Check that indeed $F'(x) = f(x)$ on both $(-\infty, 0)$ and $(0, \infty)$

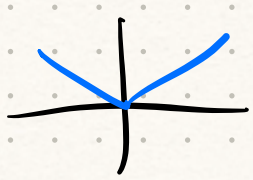
- Since this is a lot of text, we use the shorthand notation

$$\int \frac{1}{x^2} dx = -\frac{1}{x} + C$$

to denote that the functions $-\frac{1}{x} + C$ are the anti-derivatives of $f(x) = \frac{1}{x^2}$ on intervals like $(-\infty, 0)$ and $(0, \infty)$.

⚠ Point being: Always make sure you keep track of what interval you are working on! The domain of a function is part of its definition!

For another example: Think about $f(x) = |x|$



How would you talk about the indefinite integral of $f(x)$?

Worked Examples in class: let's take a few minutes to try out a few problems: Pay attention to the interval of def'n!

1) $\int (x^3 + \frac{1}{x})\,dx$

2) $\int \frac{\cos\theta}{1-\cos^2\theta}\,d\theta$

3) $\int_0^3 (x - x^{\frac{1}{2}} + 1)\,dx$

Applications of the FTC

In the last part of today's lecture, let's explore a useful reformulation of the FTC:

Recall again that if $f(x)$ is cts on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

for any anti-derivative $F(x)$ of $f(x)$ on $[a,b]$. Since $F'(x) = f(x)$ on (a,b), we can safely write

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Interpretation:

- $F(b) - F(a)$ is the net change of $F(x)$ from $x=a$ to $x=b$.
- $\int_a^b F'(x)\,dx$ is the integral of the rate of change of $F(x)$ from $x=a$ to $x=b$.

So FTC says:

The integral of the rate of change of a function is equal to the total net change of the function.

Example: Suppose a car moves along a path with velocity $v(t) = t^2 - t - 6$ km/hr at time t hrs, from $t=1$ to $t=4$.

1) What is the final displacement of the car along its path from its starting position at $t=1$?

2) How far did the car actually travel along its path from $t=1$ to $t=4$?

1) Let $S(t)$ denote a general antiderivative of $v(t)$ on $[1,4]$, so that $S'(t) = v(t)$. Then

$$\text{displacement} = S(4) - S(1) = \int_1^4 v(t)\,dt = \left(\frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t\right)\Big|_1^4$$

$$= -9/2 \text{ km}$$

2) $$\text{dist. traveled} = \int_1^4 |v(t)|\,dt = \int_1^3 (-v(t))\,dt + \int_3^4 v(t)\,dt$$

$$= 61/6 \text{ km}.$$

One last thing for today:

As an application of the FTC and the mean value formula for derivatives, we also have the following useful fact:

MVT for integrals: Let $f(x)$ be cts on $[a,b]$. Then there always exists a $c \in (a,b)$ such that $\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$.